Default Bayes Factor for Testing the (In)equality of Several Population Variances

Don van den Bergh[1⋆], Fabian Dablander[1⋆], Alexander Ly[2], Eric-Jan Wagenmakers[1]

[1]Department of Psychological Methods, University of Amsterdam, [2]Centrum Wiskunde & Informatica

Two-Sample Bayes Factor

We assume that the data from each group is normally distributed:

$$x_1 \sim \mathcal{N}\left(\mu_1,\, \sigma_1^2\right) \qquad \text{and we are interested} \qquad \mathcal{H}_0 : \sigma_1^2 = \sigma_2^2$$
$$x_2 \sim \mathcal{N}\left(\mu_2,\, \sigma_2^2\right), \qquad \text{in the hypotheses} \qquad \mathcal{H}_1 : \sigma_1^2 \neq \sigma_2^2.$$

However, testing directly whether the variances are equal complicates the choice of prior on σ_1 and σ_2. Instead, we introduce a grand variance σ^2 such that $\sigma_1^2 + \sigma_2^2 = \sigma^{2[0]}$. The variance of group one is then a fraction of the grand variance: $\rho\sigma^2 = \sigma_1^2$ and $(1-\rho)\sigma^2 = \sigma_1^2$, where $\rho \in [0, 1]$. Thus we obtain:

$$x_1 \sim \mathcal{N}\left(\mu_1,\, \rho\sigma^2\right) \qquad \text{and the hypotheses} \qquad \mathcal{H}_0 : \rho = 0.5$$
$$x_2 \sim \mathcal{N}\left(\mu_2,\, (1-\rho)\sigma^2\right), \qquad\qquad \mathcal{H}_1 : \rho \neq 0.5.$$

These hypotheses on ρ are represented by the following priors:

$$\pi\left(\mu_1,\, \mu_2,\, \sigma\right) \propto \sigma^{-2}$$
$$\pi\left(\mu_1,\, \mu_2,\, \sigma,\, \rho\right) \propto \sigma^{-2}\mathcal{B}\left(\rho;\, \alpha_1,\, \alpha_2\right).$$

These priors lead to the following Bayes factor:

$$\mathrm{BF}_{10} = \frac{\mathrm{B}\left(\frac{n_2-1}{2}+\alpha_1,\ \frac{n_1-1}{2}+\alpha_2\right)\ {}_2F_1\left(\frac{n-2}{2};\frac{n_2-1}{2}+\alpha_1;\frac{n-2}{2}+\alpha_1+\alpha_2;1-\frac{n_2 s_2^2}{n_1 s_1^2}\right)}{\mathrm{B}\left(\alpha_1,\ \alpha_2\right)\left(1+\frac{n_1 s_1^2}{n_2 s_2^2}\right)^{\frac{2-n}{2}}}.$$

Here, n_1 and n_2 are the sample sizes of each group, n is their sum, α_1 and α_2 are the parameters of the Beta prior, and s_1 and s_2 are the sample sum of squares.

Desiderata

An analytic expression for the Bayes factor is convenient, but there are more reasons to choose a $\mathcal{B}\left(\rho;\, \alpha,\, \alpha\right)$ prior as a default prior. Preliminary derivations show that the resulting Bayes factor is label invariant, measurement invariant, model selection consistent, and limit consistent.

Prior on ρ and δ

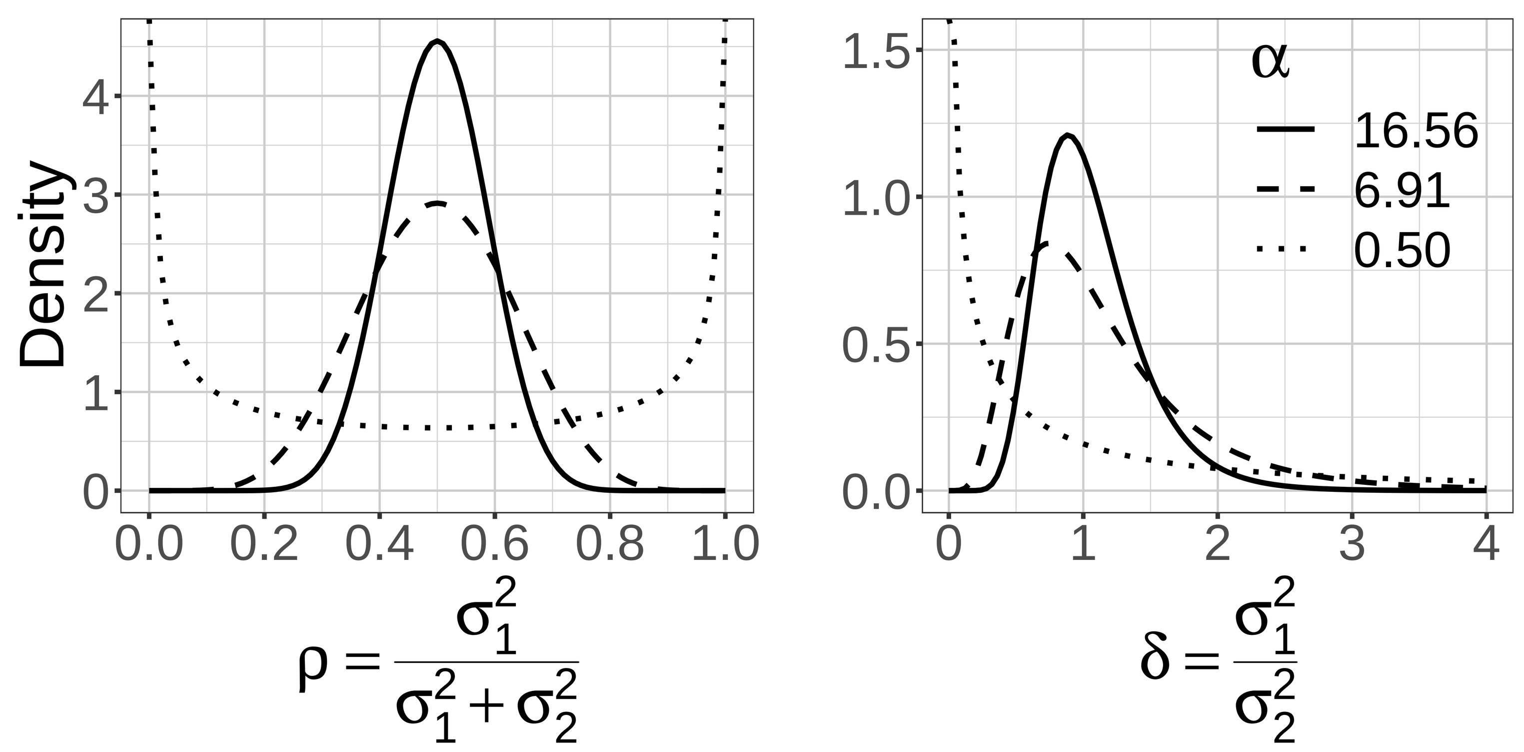

Figure 1: Left, a Beta prior on ρ which represents the fraction of the grand variance that group one makes up. Right, a Beta prime prior on the ratio of variances. The prior on δ is induced by a Beta prior on ρ.

Fractional Bayes Factor as a Special Case

Previously, a fractional Bayes factor for testing the inequality of variances was developed[1]. Interestingly, our Bayes factor appears to coincide with the fractional Bayes factor for $\alpha = 0.5$.

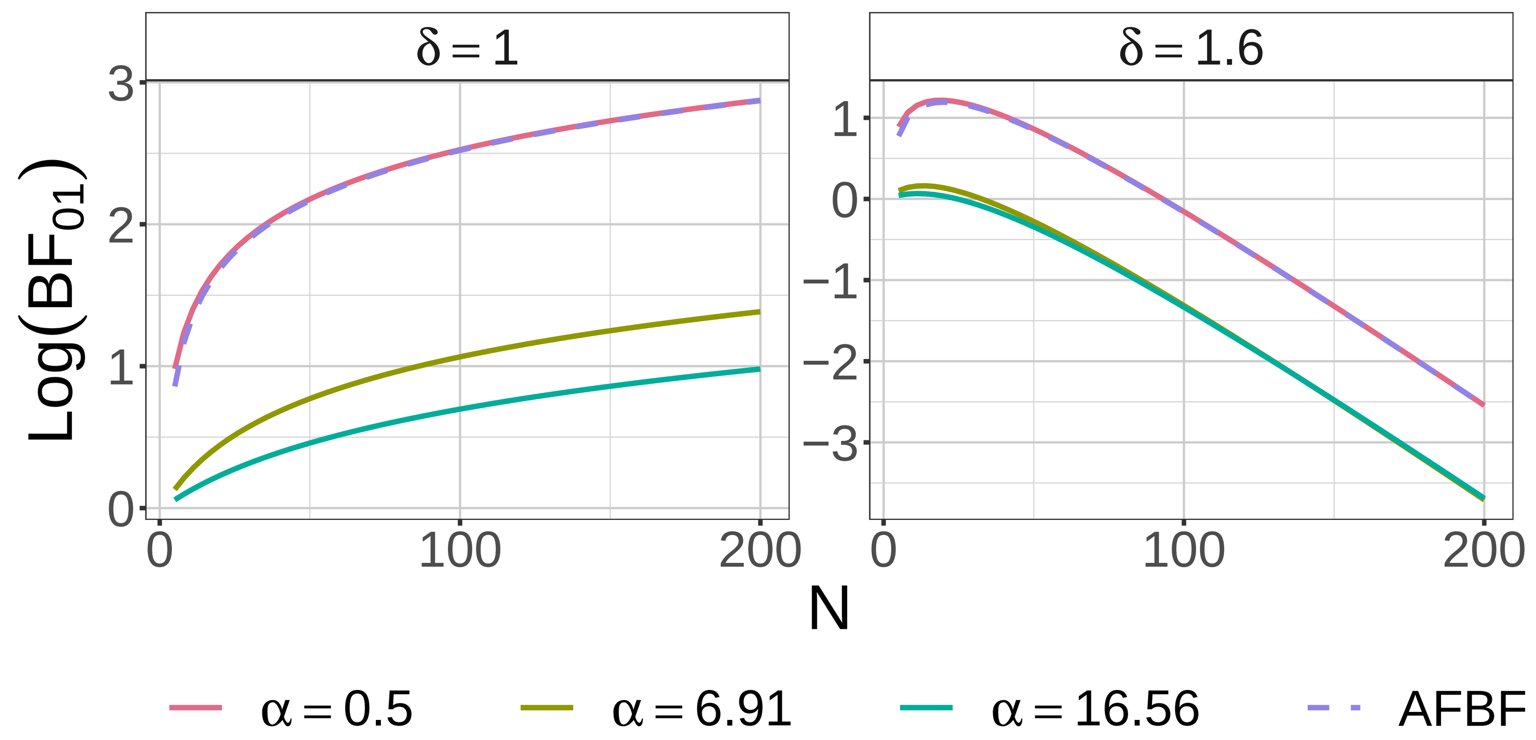

Figure 2: Comparison of a fractional Bayes factor (solid line) to our Bayes factor for various priors. The number above each panel indicates the ratio of the true variances, $\delta = \sigma_1^2/\sigma_2^2$.

One-Sample Bayes Factor

To study limit consistency for our two sample Bayes factor, we let one sample size go to infinity while keeping the other fixed. The result is the following expression:

$$\mathrm{BF}_{10}^{k=1} = \frac{1}{\mathrm{B}\left(\alpha_1,\ \alpha_2\right)}\int_0^1 (1-\rho)^{\frac{n-1}{2}+\alpha_2-1}\rho^{\alpha-\frac{1}{2}-\frac{n}{2}}\exp\left(-n\frac{s^2}{\sigma^2}\left(\frac{1}{2\rho}-1\right)\right)\ \mathrm{d}\rho.$$

Here, σ^2 is the population variance to test against while s is the sum of squares and n is the sample size of the observed data. This can be interpreted as a one-sample Bayes factor where we test the observed data against a fixed variance.

K-Sample Bayes Factor

Extending the approach to K groups is straightforward. We assume each group is normally distributed, that is $x_i \sim \mathcal{N}\left(\mu_i,\, \sigma_i^2\right)$ for $i = 1, \ldots, K$. The grand variance is defined as the sum of all variances: $\sigma^2 = \sum_{i=1}^K \sigma_i^2$. Instead of a single ρ, we now have K mixture components, such that $\sum_{i=1}^K \rho_i = 1$. Thus, we still have $\rho_i\sigma^2 = \sigma_i^2$. This implies that: $x_i \sim \mathcal{N}\left(\mu_i,\, \rho_i\sigma^2\right)$. As a prior on $\boldsymbol{\rho}$ we use a Dirichlet distribution. A multi-group situation introduces more complex hypotheses, for example:

$$\mathcal{H}_1 : \sigma_1^2 = \sigma_2^2 > \sigma_3^2, \sigma_4^2, \sigma_5^2 = \sigma_6^2 > \sigma_7^2$$

Right now, we can compute Bayes factors for such hypotheses numerically using Bridge sampling[2] but we are still working on analytical results.

Conclusion and Further Ideas

- We developed a Bayes factor for the comparison of variances that only depends on the sufficient statistics.
- Choose α in $\mathcal{B}\left(\rho;\, \alpha,\, \alpha\right)$ by relating the sufficient statistics to the critical value of a Levene's test. That would provide our Bayes factor with frequentist guarantees.
- Study the relation between the fractional Bayes factor and ours.
- Study the desiderata for the K-sample Bayes factor.

References

[0] This choice is inspired by Jeffreys's Bayes factor for the agreement of two standard errors [3, pp. 222-224].

[1] Florian Böing-Messing and Joris Mulder. Automatic Bayes factors for testing equality-and inequality-constrained hypotheses on variances. *Psychometrika*, 83(3):1–32, 2018.

[2] Quentin F Gronau, Alexandra Sarafoglou, Dora Matzke, Alexander Ly, Udo Boehm, Maarten Marsman, David S Leslie, Jonathan J Forster, Eric-Jan Wagenmakers, and Helen Steingroever. A tutorial on bridge sampling. *Journal of Mathematical Psychology*, 81:80–97, 2017.

[3] Harold Jeffreys. *Theory of Probability (1rd Ed.)*. Oxford, UK: Oxford University Press, 1939.